HW 2: 2.2 Separable eqns, slope fields

2.2: Separable Diff. Eq. continued

Entry Task: A spherical snowball melts (changes volume) at a rate proportional to its surface area. Initial the volume is 1000 in^3. $\left(r = \left(\frac{3}{4\pi}1000\right)^{1/3} = 10\left(\frac{3}{4\pi}\right)^{1/3}\right)$
Two min. later the volume is 729 in^3. $\left(r = \left(\frac{3}{4\pi}\right)^{1/3} \cdot (729)^{1/3} = 9\left(\frac{3}{4\pi}\right)^{1/3}\right)$

Solve the differential equation below (see next page for explanation of equation) and use the initial conditions, then predict how long it will take for the snowball to completely melt.

NON-LINEAR SEPARABLE

$$\frac{dV}{dt} = -DV^{\frac{2}{3}}$$

$$\int V^{-2/3}\, dV = \int -D\, dt$$

$$3V^{1/3} = -Dt + C_1$$

$$V^{1/3} = -\frac{D}{3}t + C_2 \qquad C_2 = \tfrac{1}{3}C_1, \quad D_1 = D/3$$

$$\boxed{V(t) = (-D_1 t + C_2)^3}$$

$$V(0) = 1000 \Rightarrow C_2^3 = 1000 \Rightarrow C_2 = 10$$

$$V(2) = 729 \Rightarrow (-2D_1 + 10)^3 = 729$$

$$-2D_1 + 10 = 9$$

$$-2D_1 = -1$$

$$D_1 = 1/2$$

$$\boxed{V(t) = (-\tfrac{1}{2}t + 10)^3}$$

MELT COMPLETELY

$$\Rightarrow V(t) \overset{?}{=} 0 \Rightarrow -\tfrac{1}{2}t + 10 = 0$$

$$\Rightarrow \boxed{t = 20 \text{ minutes}}$$

Set-up Notes: Note volume, radius and surface area are all changing as functions of time. Let's denote

$$V = V(t), r = r(t), S = S(t)$$

And recall: $V = \frac{4}{3}\pi r^3, S = 4\pi r^2$

And so $r = \left(\frac{3}{4\pi}\right)^{\frac{1}{3}} V^{\frac{1}{3}}$.

From the given assumption:

$$\frac{dV}{dt} = -k4\pi r^2 = -k4\pi\left(\frac{3}{4\pi}\right)^{\frac{2}{3}} V^{\frac{2}{3}}$$

Let

$$D = k4\pi\left(\frac{3}{4\pi}\right)^{\frac{2}{3}} = k(4\pi)^{\frac{1}{3}}3^{\frac{2}{3}}$$

NOTE: AS YOU SAW IN HW YOU CAN ALSO SET THIS UP IN TERMS OF r OR S. THE EASIEST IS r.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\uparrow$$

$$-k4\pi r^2 = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = -k$$

$$r(t) = -kt + C$$

$r(0) = 10\left(\frac{3}{4\pi}\right)^{1/3} \Rightarrow C = \left(\frac{3}{4\pi}\right)^{1/3}$

$r(2) = 9\left(\frac{3}{4\pi}\right)^{1/3} \Rightarrow -2k + 10\left(\frac{3}{4\pi}\right)^{1/3} = 9\left(\frac{3}{4\pi}\right)^{1/3}$

$-2k = -\left(\frac{3}{4\pi}\right)^{1/3}$

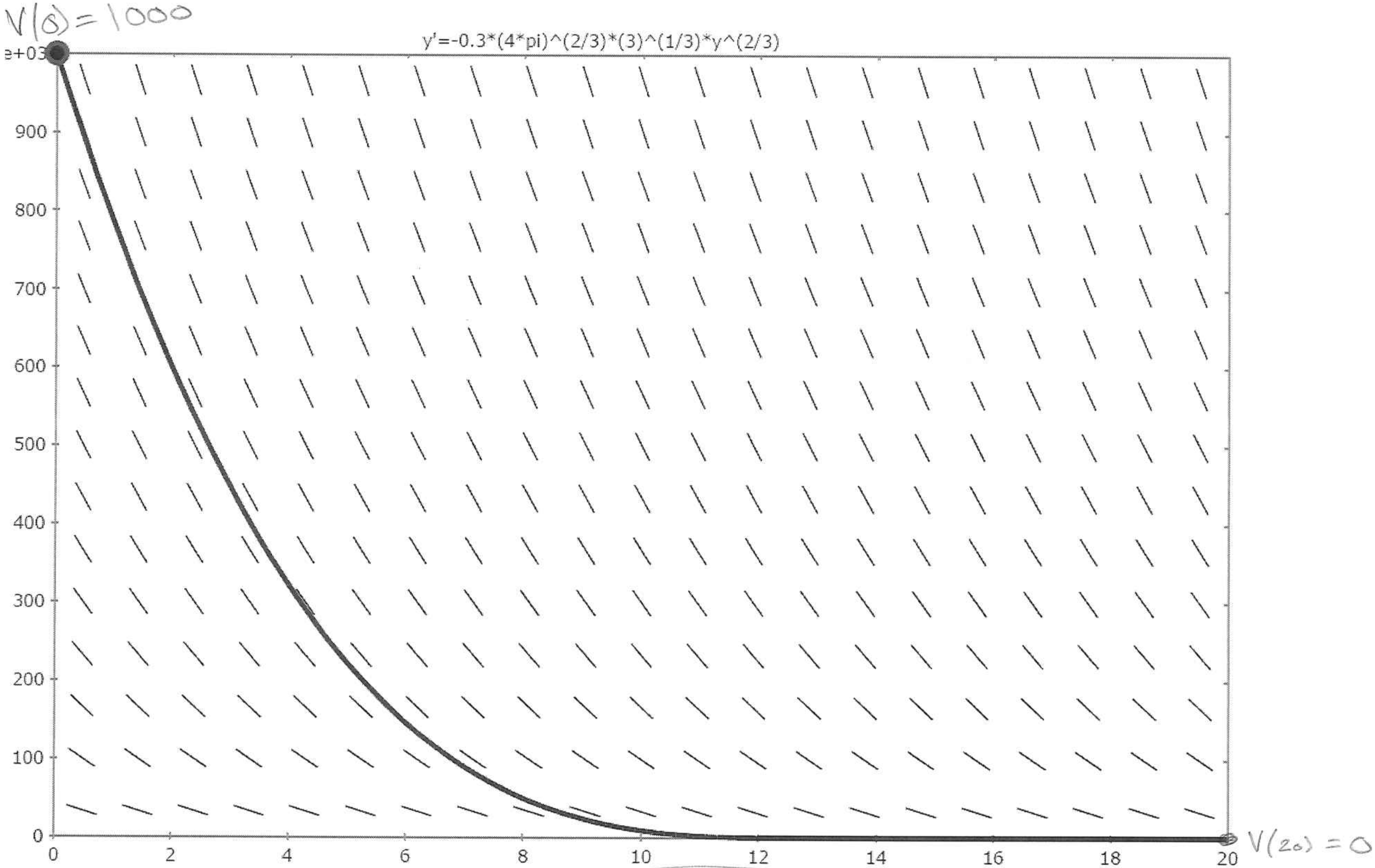
$k = \frac{1}{2}\left(\frac{3}{4\pi}\right)^{1/3} \approx 0.310135$

So

$$r(t) = -\frac{1}{2}\left(\frac{3}{4\pi}\right)^{1/3} t + 10\left(\frac{3}{4\pi}\right)^{1/3} = \left(\frac{3}{4\pi}\right)^{1/3}\left(-\frac{1}{2}t + 10\right)$$

NOTE $V(t) = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left[\left(\frac{3}{4\pi}\right)^{1/3}\left(-\frac{1}{2}t+10\right)\right]^3 = \left(-\frac{1}{2}t+10\right)^3$ WHICH MATCHES!

Slope field for the Snowball problem



SLOPE IS ALWAYS NEGATIVE IF V IS POSITIVE

Aside: $\frac{dV}{dt} = -DV^{2/3}$ $\quad 3D_1 = D$

$= -\frac{3}{2} V^{2/3}$ $\quad \uparrow$ $\frac{1}{2}$

Example:

You have \$30,000 in a bank account.

- The account earns 2% annual interest, compounded continuously.
- In addition, you withdraw money throughout the year totaling about \$1000/year.

When will you run out of money?

$A(t) =$ "amount after t yrs"

$$\frac{dA}{dt} = \overbrace{\underbrace{0.02A}_{\text{added per year}} - \underbrace{1000}_{\text{lost per year}}}^{\text{total change per yr.}}$$

$$A(0) = 30{,}000$$

$$\int \frac{1}{0.02A - 1000}\, dA = \int 1\, dt$$

$$\frac{1}{0.02} \ln|0.02A - 1000| = t + c_1$$

$$\ln|0.02A - 1000| = 0.02t + C_2 \qquad C_2 = 0.02C_1$$

$$0.02A - 1000 = \pm e^{C_2} e^{0.02t} \qquad C_3 = \pm e^{C_2}$$

$$A = \frac{1000 + C_3 e^{0.02t}}{0.02} \qquad C = \frac{C_3}{0.02}$$

$$\boxed{A(t) = 50000 + Ce^{0.02t}}$$

$$A(0) = 30{,}000 \Rightarrow C = -20{,}000$$

$$\boxed{A(t) = 50000 - 20000e^{0.02t}} \stackrel{?}{=} 0$$

$$t = \frac{\ln\left(\frac{50000}{20000}\right)}{0.02}$$

$$t \approx 45.81 \text{ years}$$

yes!

WHY MORE THAN 30 years?

How much do you need in the account initially so you never run out of money?

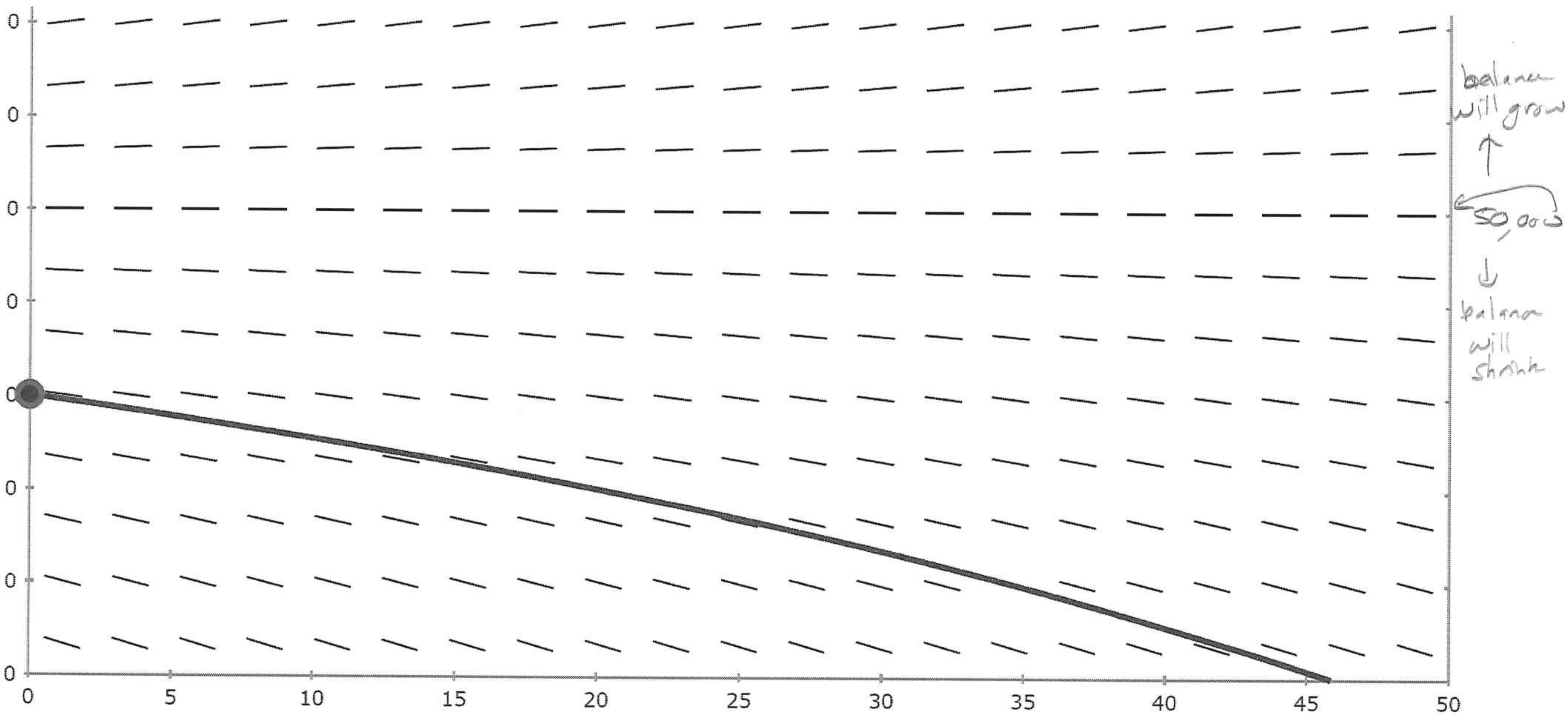
$$\frac{dA}{dt} = 0.02A - 1000 = 0 \Rightarrow A = 50{,}000$$

EQUILIBRIUM SOL'N

$$A(t) = 50{,}000$$

NOTE $C = 0$ IN THIS CASE

Slope field for Bank Account Example



Example:

Consider

$$\frac{dy}{dx} = 3x - y$$

This is NOT separable. It is "linear" and we will discuss a method on ~~Wednesday~~ for this type.

FRIDAY

But if you leave this course, you may encounter a method called "change of variable" to "fix" a problem like this. Let's try one.

Assume I tell you to let $\overset{v(x)}{v} = 3x - \overset{y(x)}{y}$

Find

$$\frac{dv}{dx} = 3 - \frac{dy}{dx} = 3 - (\overbrace{3x - y}^{v})$$

So

$$\frac{dv}{dx} = 3 - v$$

This new equation is separable!! Solve it, then rewrite your final answer in terms of y and x.

$$\int \frac{1}{3-v}\,dv = \int dx$$

$$\frac{1}{-1}\ln|3-v| = x + c_1$$

$$\ln|3-v| = -x - c_1$$

$$3 - v = \underbrace{\pm e^{-c}}_{C_2} e^{-x}$$

$C = C_2$

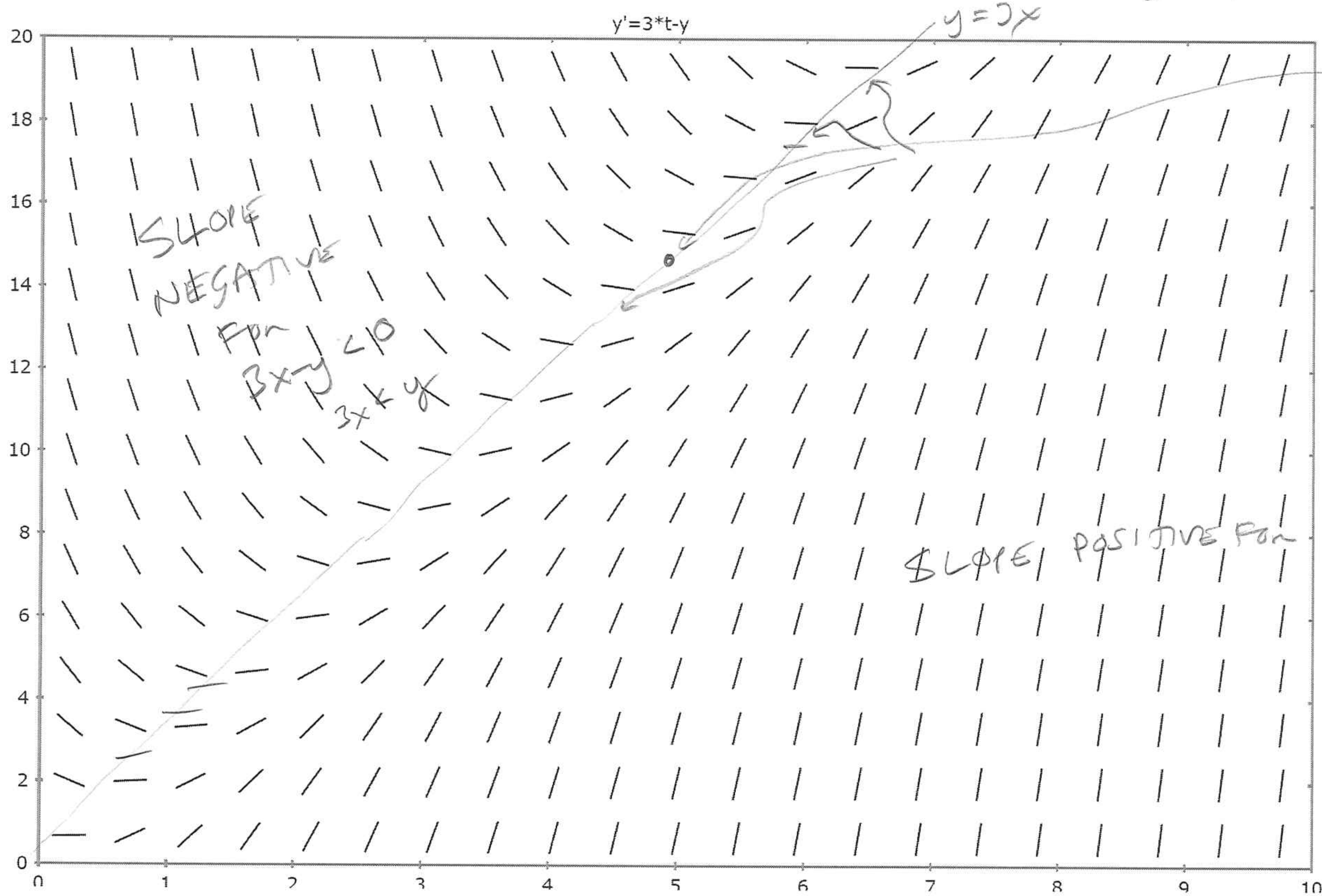
$$\boxed{v = 3 - Ce^{-x}}$$

$$3x - y = 3 - Ce^{-x}$$

$$\Rightarrow \boxed{y = 3x - 3 + Ce^{-x}}$$

Slope Field for last example

$\frac{dy}{dx} = 3x - y \overset{?}{=} 0$ WHEN $y = 3x$



SLOPE IS ZERO

$3x - y > 0$

$3x > y$